Nonchaotic attractors with highly fluctuating finite-time Lyapunov exponents

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(Received 26 November 1997)

The implications of large fluctuations of finite-time Lyapunov exponents are discussed for nonchaotic systems. We show that for a nonchaotic system driven by quasiperiodic force, if its finite-time Lyapunov exponents periodically become positive, the resulting attractor can be strange but nonchaotic. With long enough expanding time intervals, a nearly periodic force can also lead to strange nonchaotic attractors. For the case of a periodic force, a special typical periodic attractor that is sensitive to micronoise is obtained. With the different finite computing precisions, different pseudoperiodic orbits can be obtained.
[S1063-651X(98)07805-2]

PACS number(s): 05.45.+b

Strange nonchaotic attractors are attractors that are geometrically complicated, but typical trajectories on these attractors exhibit no sensitive dependence on initial conditions asymptotically. Here the word ''strange'' refers to the geometric structure of the attractor: A strange attractor is an attractor that is neither a finite set of points nor piecewise differentiable. The word chaotic refers to the dynamics of the orbits on the attractor: A chaotic attractor is one for which typical orbits have a positive Lyapunov exponent. This implies that nearby orbits diverge exponentially from one another with time and that the orbit depends sensitively on its initial conditions. These attractors were described by Grebogi *et al.* in 1984 [1].

Following the initial study were several theoretical studies pertaining to the existence and characterization of strange nonchaotic attractors [2–8]. A typical system considered in most of these works is a nonlinear continuous or discrete time oscillator with a two- or three-incommensurate-frequencies forcing. Most of these studies have focused on their characterization, either through spectral properties, geometric dimension properties, local divergence of trajectories, or their identification in a time series. Strange nonchaotic attractors can arise in physically relevant situations such as quasiperiodically forced damped pendulums [9] and localization of quantum particles in quasiperiodic potentials [10]. These exotic attractors have been observed in physical experiments [11,12].

A basic question that remains interesting is about the dynamical mechanisms responsible for the creation of strange nonchaotic attractors, i.e., what the possible routes to strange nonchaotic attractors are. Kapitaniak [13] shows an artificially controlling technique that allows us to generate a strange nonchaotic trajectory by making small changes in the parameters of the three-dimensional system. The method is applicable to the systems in which behavior depends on a control parameter c such that they have a chaotic attractor for one value of c, say c_1, and a strange repeller together with a periodic attractor for the other value of c, c_2. Here the system with a strange repeller exhibits transient chaos. One mechanism was investigated by Heagy and Hammel [14], who discovered that, in quasiperiodically driven maps, the transition from a two-frequency quasiperiodic attractor to a strange nonchaotic attractor occurs and then a period-doubled torus collides with its unstable parent torus. Near the collision, the period-doubled torus becomes extremely wrinkled and develops into a fractal set at the collision, although the Lyapunov exponent remains negative throughout the collision process. Feudel *et al.* [15] found that the collision between a stable torus and an unstable one at a dense set of points leads to a strange nonchaotic attractor. Yalcinkaya and Lai [16,17] show that for dynamical systems with an invariant subspace in which there is a quasiperiodic torus the loss of the transverse stability of the torus can lead to the birth of a strange nonchaotic attractor. A physical phenomenon accompanying this route to strange nonchaotic attractors is an extreme type of intermittency.

More recently, one of the important observations is that typical trajectories on strong nonchaotic attractors are characterized by finite-time Lyapunov exponents (or transient Lyapunov exponents) that fluctuate between positive and negative values, although asymptotically the time-independent Lyapunov exponent is negative [18,19]. Furthermore, Lai points out that [20] whether the asymptotic attractor of the system is strange nonchaotic or strange chaotic is determined by the relative weight of the contraction and expansion for infinitesimal vectors along a typical trajectory on the attractor. When the average contraction dominates the average expansion, the attractor is strange nonchaotic. The transition from strange nonchaotic to strange chaotic attractor occurs when the average contraction and expansion are balanced. A characteristic signature of this route to chaos is that the Lyapunov exponent passes through zero linearly.

Based on this research, a different route to create strange nonchaotic attractors is presented in this paper. We show that if the systems possess the following two conditions, they can

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produce strange nonchaotic attractors. One of them is that there exists the dynamics of chaotic and periodic intermittency. In other words, the finite-time Lyapunov exponents of the nonchaotic system should oscillate greatly about its negative Lyapunov exponent. As a result, one can periodically obtain positive finite-time Lyapunov exponents. A system driven by the sinusoidal force can produce this typical dynamics. This condition provides sensitive dynamics to the noise in the system. The second condition is that there should be a source to generate the necessary noise that leads the orbit to run into different diverging orbits during various expanding time intervals. A typical source is the quasiperiodical force. Under these conditions, a strange geometric structure can then be produced for the attractor. We show that there is a finite region in the parameter space for which the strange nonchaotic attractors exist. This method is quite similar to that proposed by Kapitaniak [13]. However, differently from it, the system is not switched between two states here and the periodic attractors are not required to possess the strange repellers in order to exhibit transient chaos. To obtain the chaotic and periodic intermittent dynamics, an artificially controlling technique is used in Ref. [13]. In this paper a natural sinusoidal force is applied. In contrast to the method in Refs. [16,17], the intermittency dynamics is an essential ingredient, rather than an accompanying phenomenon, to create the strange nonchaotic attractor in the present method.

To construct a quasiperiodic force, in most of the works, the golden mean $f=(\sqrt{5}-1)/2$ is used as a typical forcing frequency. In contrast, we call the quasiperiodic force with frequency $f'=n+10^{-12}\sqrt{5}$ (here n is a big enough real number, equal to 0.005) the nearly periodic one. We will show that if the finite-time Lyapunov exponents become positive with a long enough time interval, even with the nearly periodic force, one can also obtain a strange nonchaotic attractor.

If the noise source is absent, one can obtain a periodic attractor. However, in contrast to the normal ones, the periodic attractor obtained here is special; it consists of a piece of a stable period with another piece of a unstable period. If the unstable piece is long enough, it is sensitive to the micronoise. The effects of noise on chaos are discussed extensively. Here we show that some special periodic attractors also possess this property. As a result, if such systems are disturbed by micronoise, strange nonchaotic attractors can also be achieved.

In Refs. [21–25] the effects of finite computing precision leading to pseudotrajectories in chaotic systems are investigated. It is shown that in the absence of a hyperbolic structure, because the trajectories are globally sensitive to small errors, trajectories of a chaotic system will fail to have long shadowing trajectories [24]. The smaller the fluctuation of the finite-time Lyapunov exponents about zero, the shorter the shadowing distance and time [25].

The effect of the truncation error on the periodic attractor is seldom discussed. We often think that the computing precision means only the uncertainty of the periodic trajectory: higher computing precision, lower uncertainty. In fact, the truncation error can be ignored for a wholly stable attractor. However, in the paper we will show that for the periodic system, if its finite-time Lyapunov exponents fluctuate greatly about its negative Lyapunov exponent, a pseudoperiodic trajectory can be obtained. Here the term ''pseudo'' indicates that the uncertainty is not determined by the computing precision and has the order of the attractor size. The longer the time interval of the positive finite-time Lyapunov exponents, the higher the computing precision required to achieve the real periodic attractor.

To support our proposal, we present some numerical simulations on the sinusoidally driven logistic map. Consider the logistic map driven by a sinusoidal force

$$x(t+1)=[x(t)+f] \quad \bmod 1,$$

$$y(t+1)=ay(t)[1-y(t)]+A\sin[2\pi x(t)]. \tag{1}$$

There are two Lyapunov exponents for the map. One of them is always zero, while the other can simply be calculated from the tangential space of the logistic map. In the paper we set $a=3.6$, the amplitude of force $A=0.12$, and let π be on the order of 10^{-12}. Now set the frequency $f=0.005$. In this case, an exactly periodic force is used. Its Lyapunov exponent λ is -0.031. As expected, a periodic attractor is obtained, which is shown in Fig. 1(a). Its fractal dimensions are naturally equal to zero, as obtained in Fig. 2.

What will happen if we set the frequency $f'=0.005+10^{-12}\sqrt{5}$? Its corresponding Lyapunov exponent is also -0.031. Thus, at the first glance, one may think that we can obtain a periodic attractor that is very similar to that shown in Fig. 1(a). In contrast to $\sin(2\pi ft)$, the nearly periodic force $\sin(2\pi f't)$ always results in only a slightly different value, which is on the order of 10^{-12}. This microdifference can be ignored for the periodic attractors due to its negative Lyapunov exponent. However, the attractor is shown in Fig. 1(b). Surprisingly, the simulation result implies that a strange attractor is obtained here. Its capacity dimension D_C and information dimension D_I can be calculated along the line using the box-counting algorithm. If we ignore the microdifferences between various $x(t)$, only 200 discrete values of $x(t)$ are obtained. Due to the discrete x axis, we can simply count the number of one-dimensional boxes for a fixed x. With this simplification, one can easily estimate if the attractor is strange. As shown in Fig. 2, the capacity dimension $D_C=0.97$ and information dimension $D_I=0.92(\pm 0.02)$ are fitted out with $x=0.3$. This clearly indicates that the attractor possesses the strange geometric structure.

Comparing Fig. 1(a) with Fig. 1(b), the difference between the two driving forces is only on the order of 10^{-12}. It is this tiny difference that determines whether a strange attractor or a periodic attractor is achieved. This implies that the nonchaotic system is also sensitive to micronoise caused by a nearly periodic force.

From Fig. 1(b) one can clearly see that the trajectory of the attractor possesses the dynamics of periodic and chaotic intermittency. When $[x(t)] \bmod 1<0.2$ or $[x(t)] \bmod 1>0.7$, the trajectory runs into a contracting region, otherwise it runs into an expanding region. This observation indicates that, although the nonchaotic attractor has a negative Lyapunov exponent and does not yield long-term exponentially diverging solutions, its short-term dynamics can be quite complex. The short-term dynamics can be exactly de-

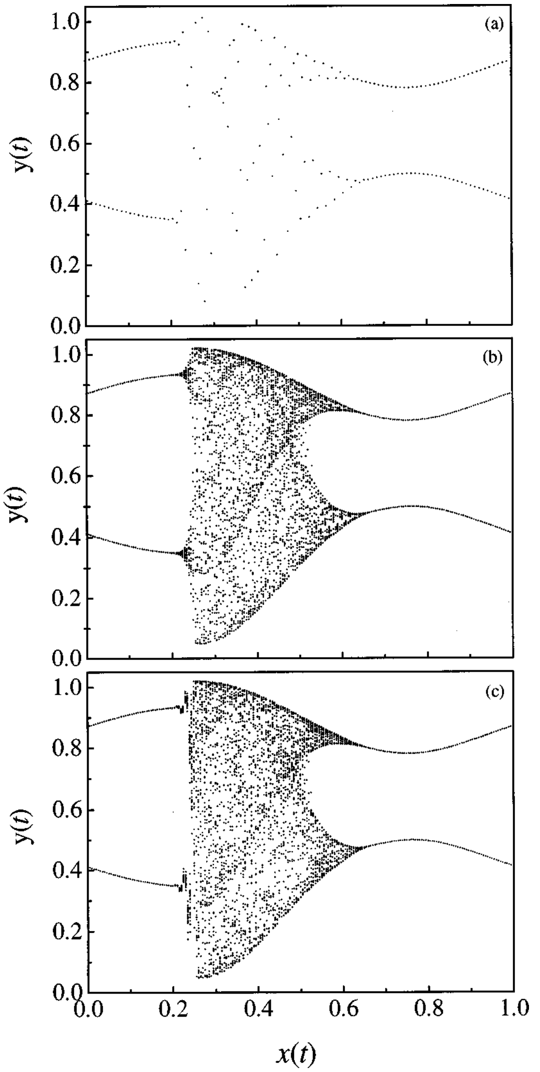

FIG. 1. Attractors of (a) Eq. (1) with $f=0.005$, (b) Eq. (1) with $f=0.005+10^{-12}\sqrt{5}$, and (c) Eq. (5) with $f=0.005$. The 20 000 pixels are dotted with $A=0.12$.

scribed by the finite-time Lyapunov exponent [18–20]. In particular, the time τ Lyapunov exponent $\lambda_\tau(t_0)$, which is defined as

$$\lambda_\tau(t_0)=\frac{1}{\tau}\sum_{t=t_0}^{t_0+\tau}\ln\left|\frac{df}{dx(t)}\right|, \tag{2}$$

quantifies the expanding or contracting exponent that the trajectory experiences in the time τ interval from time t_0. For a simple nonchaotic system with any finite τ, the time τ Lyapunov exponents are also negative or with a small fluctuation about its Lyapunov exponent. As a result, any difference among the trajectories approaches zero gradually and a wholly stable periodic attractor is achieved. The micronoise actually has little effect on the dynamics.

Now suppose that the finite-time Lyapunov exponents of the nonchaotic system fluctuate about its Lyapunov exponent greatly, so great that the positive finite-time Lyapunov exponents can be achieved periodically. The dynamics then possesses the characteristics of intermittency between periodic and chaotic states. In particular, within a finite-time interval,

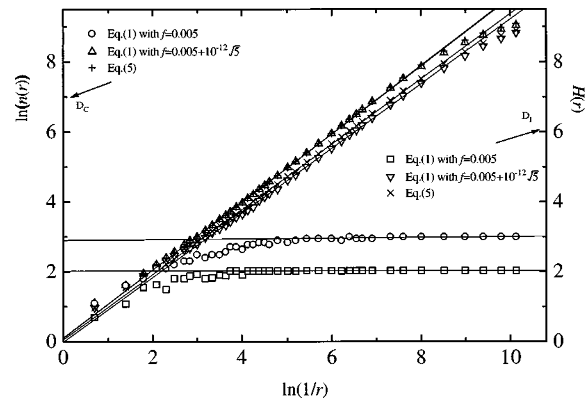

FIG. 2. Logarithm of box-counting numbers $\ln n(r)$ and information entropy $H(r)$ versus logarithmic scale $\ln(1/r)$. The data are obtained from 5×10^5 points after the initial transient 5×10^4 points have been cut with the initial condition equal to 0.2.

the trajectory visits the contracting region with a high frequency and then in the next finite-time interval it visits the expanding region with high frequency. After that, it will go back to the contracting region again and the whole process continues. If the time intervals of chaotic divergence are long enough, when a micronoise is always added to the system, it will be enlarged by the dynamics during every interval with positive finite-time Lyapunov exponents. Because of the positive finite-time Lyapunov exponents, the created orbit possesses the characteristics of chaos. As a result, a strange geometric structure is achieved.

The time τ Lyapunov exponents $\lambda_\tau(t)$ versus time t are given in Fig. 3 for Eq. (1) with $\tau=5$ and $f'=0.005+10^{-12}\sqrt{5}$. If $f=0.005$, a similar but periodic wave form is obtained. It shows that the time τ Lyapunov exponents are modulated by the sinusoidal force and have a large oscillating dynamics about its Lyapunov exponent. The modulating frequency equals the forcing frequency. When the time τ Lyapunov exponents are negative, the orbit is driven into the contracting region with high probability. On the other hand, when the time τ Lyapunov exponents are positive, the trajectory will be found in the expanding region with high probability. Within the long enough expanding time intervals, any micronoise will be enlarged exponentially and then lead to a macroeffect on the attractor.

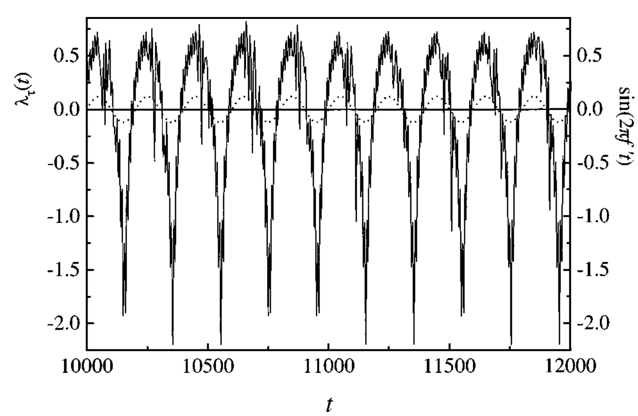

FIG. 3. Plot of time versus time τ Lyapunov exponents for Eq. (1) with $\tau=5$ and time t from 10 000 to 13 000. The dotted line shows the nearly periodic driving force.

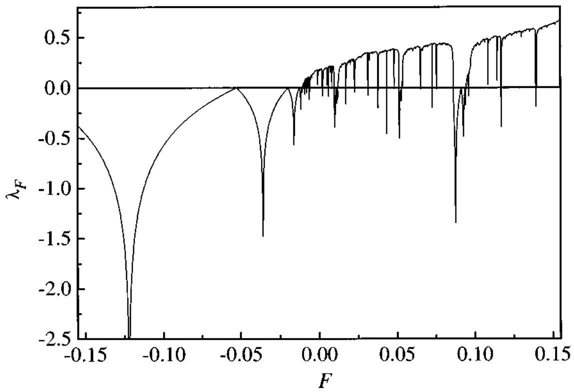

FIG. 4. Lyapunov exponents λ versus constant driving force F from -1.55 to 1.55 for Eq. (3).

However, as shown in Fig. 1(a), because of the large fluctuation of the finite-time Lyapunov exponents, the strange nonchaotic attractor will not occur. During each contracting time interval, various orbits are contracted to the same periodic orbit. Then, if starting with the same initial conditions, a single orbit will always be obtained during various expanding time intervals. This is in the case of periodic force. To obtain a strange structure, there should be a noise source to drive the same contracting orbits to run into different orbits during different expanding time intervals. In fact, the quasiperiodic force can provide such noise. Because of the small driving frequency used here, a long diverging time interval can be obtained. As a result, even with the nearly periodic force that is only on the order of 10^{-12}, the micronoise can be enlarged exponentially during various expanding time intervals and hence lead to different chaotically bursting orbits to construct a strange structure for the attractor.

Since $f \ll 1$, the driving force of the system can be approximated by a constant driving force F for a short time interval. So the time τ Lyapunov exponent $\lambda_\tau(t_0)$ at time t_0 can be approximated by the Lyapunov exponent λ_F of

$$y(t+1)=ay(t)[1-y(t)]+F, \tag{3}$$

with constant F equal to $A\ \sin(2\pi f t_0)$. For this kind of system, finite-time Lyapunov exponents have a small fluctuation about its Lyapunov exponent and so we can use this approximation. In Fig. 4 a plot of the Lyapunov exponents λ_F versus constant driving force F from -0.155 to 0.155 is given. With $F=-0.12$, $\lambda_F=-1.721$, while with $F=0.12$, $\lambda_F=0.495$, approaching the extreme values of finite-time Lyapunov exponents shown in Fig. 3. We can observe from Fig. 4 that positive Lyapunov exponents are always achieved when $F>0.118$, except in a narrow region near $F=0.1392$. On the other hand, negative values are obtained when $F<-0.011$. This means that if the amplitude of the driving force in Eq. (1) is in the region $A>0.118$, the chaotic and periodic intermittency dynamics can always be found. Simulation results show that when $0.118<A<0.15$ with $f=0.005$, the Lyapunov exponents of the attractors are all negative. This implies that there exists a finite region in the parameter space A that contains strange nonchaotic attractors. In addition, one can also always find such attractors in the region of f approaching zero. The smaller the driving frequency, the longer the expanding time interval that can be achieved and the more nearly periodic force that can be used to lead strange nonchaotic attractors. In short, the strange nonchaotic attractor is a typical phenomenon in the system, as confirmed by our computer simulations.

Now that we understand the dynamical origin of the strange nonchaotic attractor, we can construct examples in which this happens. One example is to set $f=0.002\times\sqrt{6}$ for Eq. (1). Another example is

$$y(t+1)=ay(t)[1-y(t)]+A\ \sin(2\pi f t), \tag{4}$$

with $f=0.005$ and π on the order of 10^{-12}. This is also a nearly periodic force, e.g., $\sin(2\pi)\approx 10^{-12}$. Set

$$x(t)=[ft] \quad \mathrm{mod} \quad 1.$$

The attractor obtained in the x-y phase space is very similar to that in Fig. 1(b).

In the following we will show that the periodic attractor obtained from Eq. (1) driven by periodic force possesses complex dynamical behaviors. In contrast to the normal periodic attractors, it is not a wholly stable one. It consists of a piece of a stable period with another piece of an unstable period that possesses positive finite-time Lyapunov exponents. If the time interval in which the finite-time Lyapunov exponents of the system are positive is long enough, its dynamics is more like chaos. Thus some behaviors encountered in chaotic systems can be found in such periodic systems.

The notable result is that the unstable periodic piece will show strong sensitivity to micronoise, which in turn leads to macroeffects in the chaotic divergent intervals. As the added micronoise can produce various diverging orbits during different expanding time intervals, another route to achieve the strange nonchaotic attractor can be developed. It is produced with a periodic force disturbed by micronoise,

$$x(t+1)=[x(t)+f] \quad \mathrm{mod} \quad 1,$$

$$y(t+1)=ay(t)[1-y(t)]+A\ \sin[2\pi x(t)]+\delta_{\mathrm{noise}}. \tag{5}$$

Suppose that the amplitude of the micronoise is on the order of 10^{-12}. The corresponding attractor is shown in Fig. 1(c). Statistically, it is similar to the strange nonchaotic attractor caused by Eq. (1). Its fractal dimensions are calculated from Fig. 2 as $D_C=0.97$ and $D_I=0.93$ (± 0.02) with $x=0.3$, which is almost the same as the results of Fig. 1(b).

Now if we construct such an experimental system, unlike usual periodic systems, the system's predictability also oscillates with time because experimental noise cannot be avoided. In particular, one cannot predict the orbit during its chaotic divergent intervals. Another consequence is that it seems that we cannot distinguish system (1) from system (5) in experiment.

As truncation error is a special kind of noise, one can easily deduce from the above analysis that the truncation error also causes some basic problems in such systems. Because of the absence of a noise source, the periodic attractor is also obtained using the float precision computation. However, this periodic orbit $y'(t)$ is different from the orbit $y(t)$ obtained with double precision computation. A plot of time versus the differences $\Delta y(t)=y'(t)-y(t)$ within a driving period is given in Fig. 5. With the same precision of compu-

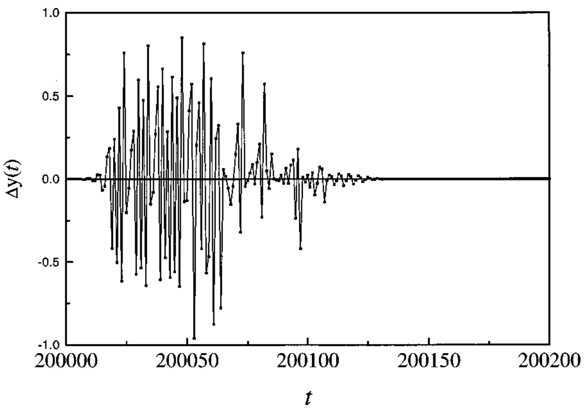

FIG. 5. Plot of time versus the differences $\Delta y(t)$ between the orbit obtained with 10^{-9} truncation error and that with 10^{-18}.

tation, the absence of noise source results because the change in truncation errors is periodic with the same frequency as the driving force. Different finite precisions lead to different orbits of periodic truncation errors, which in turn cause different pseudoperiodic attractors. In this case, the uncertainty of the periodic trajectory is not proportional to the truncation error. The error between the computing trajectory and the real trajectory can be of the order of the attractor size. So, for the periodic system, if its finite-time Lyapunov exponents fluctuate greatly about its negative Lyapunov exponent, a pseudoperiodic trajectory can often be obtained. With much higher computing precision, the noise caused by the truncation error can be ignored. In this case, the real periodic attractor is obtained. The smaller the driving frequency f, the higher the computing precision needed to achieve the real periodic attractor.

The implications of large fluctuations of finite-time Lyapunov exponents are discussed for nonchaotic systems. For this purpose, the logistic map driven by a sinusoidal force is considered. Our research indicates that for systems with finite-time Lyapunov exponents fluctuating greatly about their Lyapunov exponent, i.e., with the dynamics of chaotic and periodic intermittency, the behaviors of the chaos and period can be achieved in turn. As a result, the attractor corresponds to the coexistence of the pieces of stable and unstable structures. The unstable structure is strange if the system is driven by a quasiperiodic force or by a periodic force disturbed by noise. The lower the driving frequency, the more the nearly periodic force can be used. For the periodic forcing, a special typical periodic attractor can be obtained that is sensitive to the micronoise. With the different finite computing precisions, different pseudoperiodic orbits can be obtained.

To construct such a system it is important to find a suitable equation. The equation should be chaotic (or periodic) when it is disturbed by a constant force F (or $-F$). The second condition is that the equation disturbed by a sinusoidal force with amplitude F is still nonchaotic. Here the forcing frequency should be small enough. In this case, when the sinusoidal force approaches F, it can be approximated by the constant force F for a long enough time. As a result, positive finite-time Lyapunov exponents can be periodically achieved.

The authors would like to thank the City University of Hong Kong for providing the Strategic Research Grant for this work.

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